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HARRIS OPERATORS

BY

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ABSTRACT

A method is constructed which leads to a proof for both the "zero-two" law, and the Ornstein–Métivier–Brunel Theorem for Harris operators. For the proof it is not necessary to assume that the measure space is measurable and the operator need not be given by a transition probability. We strove to make these notes self-contained.

In these notes we attempt to describe, in a self-contained fashion, the theory of Harris operators. In particular we shall prove here the "Ornstein–Métivier–Brunel Theorem" and the "zero-two" law.

Since the notes are intended for the nonspecialist we shall not assume any knowledge of the theory of Markov operators but will prove the necessary results. Thus only measure theory and elementary functional analysis are used. One exception though — we shall use a classical result on the existence of an invariant measure.

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In the preparation of Section VIII I was lucky to benefit from many conversations with Nassif Ghoussoub.

The notes are dedicated to the memory of Shlomo Horowitz, who was my student and my colleague and whose research added many original and elegant results to this theory.

I. Definitions and notation

Let (X, Σ, λ) be a measure space and $\lambda(X) = 1$.

We shall study $L_\infty(X, \Sigma, \lambda)$. Thus every relation will be in the "a.e." sense unless otherwise stated. Every function is assumed to be measurable, every set is

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assumed to be in Σ and every signed measure (or σ finite measure) is assumed to be weaker than λ.

DEFINITION 1.1. A Markov operator is a linear operator P, on $L_\infty(X,\Sigma,\lambda)$ such that

(1) if $f \geqq 0$ then $Pf \geqq 0$,

(2) $P1 \leqq 1$,

(3) if $f_n \downarrow 0$ then $Pf_n \to 0$.

The operator P acts on signed measures by

$$\mu P(A) = \int P1_A d\lambda,$$

where 1_A is the characteristic function of A.

It is easy to see that μP is again a signed measure weaker than λ. Use the Radon–Nikodym Theorem to define uP by:

$$\textit{if } d\mu = ud\lambda \textit{ then } d(\mu P) = (uP)d\lambda.$$

This operator on $L_1(X,\Sigma,\lambda)$ satisfies:

(i) *If* $u \in L_1$ *and* $f \in L_\infty$ *then* $\int (uP)fd\lambda = \int u(Pf)d\lambda$.

(ii) *If* $u \geqq 0$ *then* $uP \geqq 0$.

(iii) $\int |uP|\,d\lambda \leqq \int |u|\,d\lambda$.

To see (iii) let $u = u^+ - u^-$; then

$$\int |uP|\,d\lambda \leqq \int (u^+P + u^-P)d\lambda = \int u^+(P1)d\lambda + \int u^-(P1)d\lambda$$

$$\leqq \int (u^+ + u^-)d\lambda = \int |u|\,d\lambda.$$

It is easy to see that (i), (ii) and (iii) imply (1), (2) and (3) of Definition 1.1 if P is defined as the adjoint operator.

The operator P may be extended uniquely to all nonnegative measurable functions by:

$$\textit{if } f_n \in L_\infty \textit{ and } f_n \uparrow f \textit{ put } Pf = \lim Pf_n,$$

$$\textit{if } u_n \in L_1 \textit{ and } u_n \uparrow u \textit{ put } uP = \lim u P_n.$$

References. The study of Markov operators was initiated by E. Hopf in [10]. A more detailed discussion of the above notions is given in chapter I of [5].

II. The Hopf decomposition into conservative and dissipative parts

Let P be a Markov operator.

DEFINITION 2.1.

$$\Omega = \{f: 0 \leqq f \leqq 1 \text{ and } Pf \leqq f \text{ and } \lim P^n f = 0\},$$

$$D = \bigcup_{f\in\Omega} \{x: f(x) > 0\} = \sup\{1_{f>0}: f \in \Omega\},$$

$$C = X - D.$$

The sup here is in the L_∞ (a.e.) sense: Every bounded collection has a least upper bound and it is the supremum of a countable subcollection, see [16] proposition II.4.1.

Let $f \in \Omega$, then $P^k f \leqq f$, thus $\Sigma_{n=0}^{N} P^n(f - P^k f) \leqq k$ and the same inequality holds for the infinite sum. Put $A = \{x: f(x) - P^k f(x) \geqq \varepsilon\}$, then $1_A \leqq \varepsilon^{-1}(f - P^k f)$, hence $\Sigma_{n=0}^{\infty} P^n 1_A \leqq \text{const} < \infty$. As $k \to \infty$ and $\varepsilon \to 0$ the set A converges to $\{x: f(x) > 0\}$. Since D is a countable union of such sets it follows that:

THEOREM 2.1. *$D = \bigcup_{k=1}^{\infty} D_k$ where $\Sigma_{n=0}^{\infty} P^n 1_{D_k}$ is bounded.*

If $0 \leqq u \in L_1$ then

$$\int \left(\sum_{n=0}^{\infty} uP^n\right) 1_{D_k} d\lambda = \int u \left(\sum_{n=0}^{\infty} P^n 1_{D_k}\right) d\lambda < \infty.$$

Hence $\Sigma_{n=0}^{\infty} uP^n$ is finite on D_k for every k, thus:

THEOREM 2.2. *If $u \in L_1$ then $\Sigma_{n=0}^{\infty} uP^n$ is finite on D.*

Let us consider the set C now.

LEMMA 2.3. *Let $0 \leqq f \in L_\infty$ be such that $\Sigma_{n=0}^{\infty} P^n f \leqq K < \infty$ then $\Sigma_{n=0}^{\infty} P^n f$ vanishes on C.*

PROOF. $K^{-1} \Sigma_{n=0}^{\infty} P^n f \in \Omega$ and, by definition, vanishes on C.

LEMMA 2.4. *Let $0 \leqq f \in L_\infty$ and $Pf \leqq f$, then $Pf(x) = f(x)$ if $x \in C$.*

PROOF. The function $f - Pf$ satisfies the condition of the previous lemma, hence vanishes on C.

Let us improve these two lemmas.

THEOREM 2.5. *Let* $f \geqq 0$ *then* $\Sigma_{n=0}^{\infty} P^n f$ *assumes the values zero or infinity, only, on* C.

PROOF. Put $h = \min(1, \Sigma_{n=0}^{\infty} P^n f)$ then $0 \leqq h \leqq 1$, $Ph \leqq h$ and $P^n h(x) \to 0$ whenever $\Sigma_{n=0}^{\infty} P^n f(x) < \infty$. But, by Lemma 2.4, $h(x) = Ph(x) = \cdots = P^n h(x)$ if $x \in C$. Thus, if $x \in C$ then either $\Sigma_{n=0}^{\infty} P^n f(x) = \infty$ or $h(x) = 0$ in which case $\Sigma_{n=0}^{\infty} P^n f(x) = 0$ too.

THEOREM 2.6. *Let* $0 \leqq f < \infty$ *and* $Pf \leqq f$. *Then* $Pf(x) = f(x)$ *if* $x \in C$.

PROOF. $\Sigma_{n=0}^{\infty} P^n (f - Pf) \leqq f < \infty$ thus, by Theorem 2.5, the sum vanishes if $x \in C$: on C, $f = Pf$.

COROLLARY. *If* $x \in C$ *then* $P1(x) = 1$.

DEFINITION 2.2. The operator P is called conservative if $X = C (D = \emptyset)$.

THEOREM 2.7. *P is conservative if and only if*: $0 \leqq f \leqq 1$ *and* $Pf \leqq f$ *implies* $Pf = f$.

PROOF. If P is conservative use Theorem 2.6, if $D \neq \emptyset$ choose a nonzero function in Ω.

COROLLARY. *If P is conservative then* $P1 = 1$ *and* P^k *is conservative too.*

PROOF. The first part follows from Theorem 2.7. Now let $0 \leqq f \in L_\infty$ with $0 \leqq (I - P^k) f$, then $0 \leqq (I - P)(I + P + \cdots + P^{k-1}) f$ and equality holds by Theorem 2.7.

THEOREM 2.8. *Let P be a conservative operator and* $f \geqq 0$ *and* $Pf = f$. *Then* $P 1_{\{x : f(x) > a\}} = 1_{\{x : f(x) > a\}}$.

PROOF.

$$f - a = (f - a)^+ - (f - a)^- = P[(f - a)^+] - P[(f - a)^-].$$

Thus $P[(f - a)^-] \geqq (f - a)^-$ and $(f - a)^- \leqq a$. Apply Theorem 2.7 to $a - (f - a)^-$ to conclude that $P[(f - a)^-] = (f - a)^-$. Therefore $P[(f - a)^+] = (f - a)^+$ too. Thus $P[\min(1, n(f - a)^+)] \leqq \min(1, n(f - a)^+)$ and, again, equality holds. Let $n \to \infty$ to obtain the result.

DEFINITION 2.3. The operator P is called ergodic if $P 1_A = 1_A$ implies $\lambda(A)(1 - \lambda(A)) = 0$.

COROLLARY. *Let P be ergodic and conservative. If* $f \geqq 0$ *and* $Pf \leqq f$ *then* $f = \text{const}$.

Another characterization of ergodic and conservative operators is given by:

THEOREM 2.9. *Let P be ergodic and conservative and $f \geqq 0$, but not identically zero, then* $\sum_{n=0}^{\infty} P^n f \equiv \infty$.

PROOF. It is enough to prove the result when $f = 1_A$ where $\lambda(A) > 0$. Put

$$A_N = \left\{x : \sum_{n=0}^{N} P^n 1_A(x) \geqq \frac{1}{N}\right\}, \qquad A_\infty = \left\{x : \sum_{n=0}^{\infty} P^n 1_A(x) > 0\right\}.$$

Then $A_N \uparrow A_\infty$ and

$$1_{A_N} \leqq N \sum_{n=0}^{N} P^n 1_A.$$

Thus $P1_{A_N}(x) = 0$ if $x \notin A_\infty$ or $P1_{A_N} \leqq 1_{A_\infty}$. Let $N \to \infty$ to conclude $P1_{A_\infty} = 1_{A_\infty}$ thus, by the Corollary to Theorem 2.8, $A_\infty = X$. Finally $X = C$ so if $\sum_{n=0}^{\infty} P^n 1_A(x) > 0$ then $\sum_{n=0}^{\infty} P^n 1_A(x) = \infty$.

References. The results described in this section are all classical, most proved in [10], see also [16] and [5]. This presentation is different since the Hopf Maximal Ergodic Lemma was not used.

III. The definition of a cycle

Throughout this section we shall use

ASSUMPTION 3.1. $P1 = 1$, *and if* $f \geqq 0$ *and* $Pf \equiv 0$, *then* $f \equiv 0$.

Note that if P is conservative then Assumption 3.1 holds.
If $Pf \equiv 0$ then $\sum_{n=0}^{\infty} P^n f = f < \infty$, so the sum is zero or $f \equiv 0$.

LEMMA 3.1. *Let* $P1_{A_1} = 1_{B_1}$ *and* $P1_{A_2} = 1_{B_2}$, *then* $P1_{A_1 \cup A_2} = 1_{B_1 \cup B_2}$.

PROOF.

$$\begin{aligned} 1_{B_1} + 1_{B_2} = P(1_{A_1} + 1_{A_2}) \geqq P1_{A_1 \cup A_2} = P(\max(1_{A_1}, 1_{A_2})) \\ \geqq \max(P1_{A_1}, P1_{A_2}) = \max(1_{B_1}, 1_{B_2}) = 1_{B_1 \cup B_2}. \end{aligned}$$

Thus if $x \in B_1 \cup B_2$ then $1 = 1_{B_1 \cup B_2}(x) \leqq P1_{A_1 \cup A_2}(x) \leqq 1$. On the other hand if $x \notin B_1 \cup B_2$ then $0 \leqq P1_{A_1 \cup A_2}(x) \leqq 1_{B_1}(x) + 1_{B_2}(x) = 0$.

LEMMA 3.2. *Let P satisfy Assumption* 3.1 *and let* $P1_{A_1} = 1_{B_1}$ *and* $P1_{A_2} = 1_{B_2}$. *If* $B_1 \subset B_2$ *then* $A_1 \subset A_2$.

PROOF. $P1_{A_1 \cup A_2} = 1_{B_1 \cup B_2} = 1_{B_2} = P1_{A_2}$. Thus

$$0 = P1_{A_1 \cup A_2} - P1_{A_2} = P1_{A_1 \cup A_2 - A_2}.$$

Hence, by Assumption 3.1, $A_1 \cup A_2 = A_2$.

LEMMA 3.3. *Let P satisfy Assumption 3.1. If $0 \leqq f \leqq 1$ and $Pf = 1_B$ then $f = 1_{\{x: f(x)>0\}}$.*

PROOF. Put $A = \{x: f(x) \geqq a\}$ for some $a > 0$. Then $f \geqq a^{-1}1_A$ and $1_B = Pf \geqq a^{-1}P1_A$ or $P1_A(x) = 0$ if $x \in B'$. Thus $P1_A \leqq 1_B$. Let $a \to 0$ to conclude $1_B \geqq P1_{\{x: f(x)>0\}} \geqq Pf = 1_B$.

Let P be an ergodic and conservative operator. The operator P^k is conservative again but may fail to be ergodic. Put

$$\theta = \{A: P^k 1_A = 1_A\}.$$

By Lemma 3.1 θ is a σ subfield of Σ. If $A \in \theta$ then

$$0 = (I - P)(I + P + \cdots + P^{k-1})1_A$$

hence $(I + P + \cdots + P^{k-1})1_A = \text{const}$ or $(I + P + \cdots + P^{k-1})1_A \geqq 1$. This implies that θ is atomic: otherwise we may find a sequence $A_n \in \theta$ where $A_n \downarrow$ and $\lambda(A_n) \to 0$ thus part (3) of Definition 1.1 is violated. Let B_0 be an atom of θ. By Lemma 3.3, $P^r 1_{B_0}$ is again a characteristic function. Put $P^r 1_{B_0} = 1_{B_r}$, $0 \leqq r < k$. By Lemma 3.3, B_r is again an atom of θ.

THEOREM 3.4. *Let P be an ergodic and conservative operator. Given an integer k there exist sets $B_0, B_1, \cdots, B_{d-1}$ where $d \mid k$, the sets are disjoint, $\bigcup_{i=0}^{d-1} B_i = X$ and $P1_{B_i} = 1_{B_{i+1}}$ where $B_d = B_0$. If $P^k 1_A = 1_A$ then A is the union of some of the sets B_i.*

PROOF. Define B_r as above and let d be the smallest integer for which $P^d 1_{B_0} = 1_{B_0}$. If $0 \leqq i < j < d$ and $B_i = B_j$ then $1_{B_0} = P^{d-j}1_{B_j} = P^{d-(j-i)}1_{B_0}$, a contradiction. Now $d \mid k$ and $\Sigma_{i=0}^{d-1} 1_{B_i}$ is invariant, hence it is identicaly one. Note that $B_i \cap B_j = \emptyset$ since they are atoms. Now if $A \in \theta$ then $A \cap B_i$ is either B_i or empty since B_i is an atom.

Let P be an ergodic and conservative operator and put

DEFINITION 3.1. $\Sigma_n = \{A: P^n 1_A$ is a characteristic function$\}$.

Then

(a) Σ_n is a σ subfield of Σ: Lemma 3.1.

(b) $\Sigma_n \supset \Sigma_{n+1}$: Lemma 3.3.

DEFINITION 3.2. $\Sigma^{(1)} = \bigcap_{n=1}^{\infty} \Sigma_n$.

Again $\Sigma^{(1)}$ is a σ subfield of Σ. Let $A \in \Sigma^{(1)}$ and $1_B = P1_A$. Then $P^k 1_B = P^{k+1} 1_A$ is a characteristic function, hence $B \in \Sigma^{(1)}$ too. By an obvious abuse of language we shall write $PA = B$. Now we saw

$$\Sigma^{(1)} \supset P\Sigma^{(1)} \supset P^2\Sigma^{(1)} \supset \cdots.$$

DEFINITION 3.3. $\Sigma^{(2)} = \bigcap_{n=0}^{\infty} P^n \Sigma^{(1)}$.

By Lemma 3.1 $P^k\Sigma^{(1)}$ is a field, hence so is $\Sigma^{(2)}$. Now if $B_n \in P^k\Sigma^{(1)}$ and $B_n \uparrow B$ then $1_{B_n} = P^k 1_{A_n}$ and, by Lemma 3.2, $A_n \uparrow A$ thus $P^k\Sigma^{(1)}$ is a σ field and so is $\Sigma^{(2)}$.

Let us see how P acts on $\Sigma^{(2)}$: If $A \in \Sigma^{(2)}$ and $B = PA$ then $A \in P^k\Sigma^{(1)}$, hence $B \in P^{k+1}\Sigma^{(1)}$ for all k, thus $B \in \Sigma^{(2)}$.

Again let $A \in \Sigma^{(2)}$, then $A \in P^{k+1}\Sigma^{(1)}$ or $1_A = P(P^k 1_{E_k})$ where $E_k \in \Sigma^{(1)}$. By Lemma 3.3 $P^k 1_{E_k}$ is a characteristic function. By Lemma 3.2 $P^k 1_{E_k} = 1_E$ is independent of k. Thus $E \in \Sigma^{(2)}$ and $1_A = P1_E$ or $P\Sigma^{(2)} = \Sigma^{(2)}$. Let us summarize.

THEOREM 3.5. *Let P be an ergodic and conservative operator, then $\Sigma^{(2)}$ is a σ subfield of Σ which is mapped by P onto itself.*

Later we shall prove that if P is a Harris operator then $\Sigma^{(1)}$, and thus $\Sigma^{(2)}$ too, is atomic (Lemma 5.3). This motivates the next result.

THEOREM 3.6. *Let P be ergodic and conservative. If $\Sigma^{(2)}$ is atomic then $\Sigma^{(2)} = \{A_0, A_1, \cdots, A_{d-1}\}$ where A_i are disjoint, $\bigcup_{i=0}^{d-1} A_i = X$ and $P1_{A_i} = 1_{A_{i+1}}$ where $A_d = A_0$.*

PROOF. Let A_0 be an atom of $\Sigma^{(2)}$ and put $A_i = P^i A_0$. Since P is an automorphism of $\Sigma^{(2)}$ onto itself the sets A_i are atoms too. We cannot have them all disjoint since this would imply $\sum_{i=0}^{\infty} P^i 1_{A_0} \leqq 1$ contradicting conservativeness. If $P^i A_0 = P^{i+k} A_0$ then, by Lemma 3.2, $P^k A_0 = A_0$. Let d be the smallest integer for which $P^d A_0 = A_0$. Then $\sum_{i=0}^{d-1} 1_{A_i}$ is invariant hence identically one. Finally, if $A \in \Sigma^{(2)}$ then $A \cap A_i$ is either empty or A_i.

COROLLARY. *Let P be conservative and $\Sigma^{(2)}$ be atomic. $\Sigma^{(2)}$ is trivial if and only if P^k is ergodic for every k.*

PROOF. If P^d is ergodic then $A_0 = X$. If P^k is not ergodic, for some k, then, by Theorem 3.4, $\Sigma^{(2)}$ is not trivial.

The decompositions described in Theorem 3.5 and Theorem 3.6 are called cycles.

If $\Sigma^{(2)}$ is atomic then the restriction of P^d to A_i has ergodic powers and the Corollary applies.

References. Similar notions are discussed in [13], [15] and [18].

IV. Convergence of the iterates

The collection of Markov operators is ordered:

$$P_1 \leqq P_2 \text{ if } P_1 f \leqq P_2 f \text{ for all } 0 \leqq f \in L_\infty.$$

Let us use

DEFINITION 4.1. For every $0 \leqq f \in L_\infty$

$$(P_1 \wedge P_2)(f) = \inf\{P_1 g + P_2(f-g) \colon 0 \leqq g \leqq f\}.$$

It is clear that $P_1 \wedge P_2 \leqq P_1$ and $P_1 \wedge P_2 \leqq P_2$: choose $g = f$ or $g = 0$.

If Q is a Markov operator and $Q \leqq P_1$, $Q \leqq P_2$ then, for each $0 \leqq g \leqq f$, $Qf = Qg + Q(f-g) \leqq P_1 g + P_2(f-g)$. Thus $Q \leqq P_1 \wedge P_2$. Let us establish that $P_1 \wedge P_2$ is additive for nonnegative functions. This will show that it can be extended to a linear operator on L_∞. Let $0 \leqq f_1, f_2 \in L_\infty$ and $0 \leqq g \leqq f_1 + f_2$. Put $g_1 = \min(g, f_1)$ and $g_2 = g - g_1$, then $0 \leqq g_1 \leqq f_1$ and $0 \leqq g_2 \leqq f_2$: if $g_1(x) = g(x)$ then $g_2(x) = 0 \leqq f_2(x)$. If $g_1(x) = f_1(x)$ then $0 \leqq g(x) - f_1(x) = g_2(x) \leqq f_2(x)$.

Additivity is now immediate.

Later we shall study

DEFINITION 4.2. For every $0 \leqq f \in L_\infty$, $(P_1 \vee P_2)(f) = \sup\{P_1 g + P_2(f-g) \colon 0 \leqq g \leqq f\}$.

This is the smallest linear operator which is greater than both P_1 and P_2. It may fail to be a Markov operator since $(P_1 \vee P_2)1$ may be greater than 1 at some points.

ASSUMPTION 4.1. *Let P, Q_1 and Q_2 be commuting Markov operators such that*

(a) $P1 = Q_1 1 = Q_2 1 = 1$.

(b) *There exist integers r_i and Markov operators R_i such that*

$$P^{r_i} \geqq R_i Q_1 \quad \text{and} \quad P^{r_i} \geqq R_i Q_2.$$

(c) $R_1 \cdots R_n \neq 0$ *for all* n.

From (b) follows

$$P^{r_i} = R_i Q_1 + S_i' = R_i Q_2 + S_i'' = R_i \tfrac{1}{2}(Q_1 + Q_2) + \tilde{S}_i$$

where S'_i, S''_i and $\tilde{S}_i$ are all nonnegative.

Let us prove, by induction, that

$$P^{r_1+\cdots+r_n} = R_1\cdots R_n \frac{1}{2^n}(Q_1+Q_2)^n + S_n; \qquad S_n \geqq 0 \tag{i}$$

If $n=1$ take $S_1=\tilde{S}_1$. Now, induct

$$\begin{aligned}
P^{r_1+\cdots+r_n+r_{n+1}} &= R_1\cdots R_n \frac{1}{2^n}(Q_1+Q_2)^n P^{r_{n+1}} + S_n P^{r_{n+1}}\\
&= R_1\cdots R_n P^{r_{n+1}} \frac{1}{2^n}(Q_1+Q_2)^n + S_n P^{r_{n+1}}\\
&= R_1\cdots R_n\left[R_{n+1}\frac{1}{2}(Q_1+Q_2)+\tilde{S}_{n+1}\right]\frac{1}{2^n}(Q_1+Q_2)^n\\
&\qquad + S_n P^{r_{n+1}}\\
&= R_1\cdots R_n R_{n+1}\frac{1}{2^{n+1}}(Q_1+Q_2)^{n+1} + S_{n+1}
\end{aligned}$$

where

$$S_{n+1} = R_1\cdots R_n\tilde{S}_{n+1}\frac{1}{2^n}(Q_1+Q_2)^n + S_n P^{r_{n+1}} \geqq 0. \tag{ii}$$

Equation (i) may be improved to

$$P^{(r_1+\cdots+r_n)j} = T_j\frac{1}{2^n}(Q_1+Q_2)^n + S^j_n; \qquad T_j \geqq 0. \tag{iii}$$

If $j=1$ take $T_1 = R_1\cdots R_n$ and use (i). Induct

$$\begin{aligned}
P^{(r_1+\cdots+r_n)(j+1)} &= T_j\frac{1}{2^n}(Q_1+Q_2)^n P^{r_1+\cdots+r_n} + S^j_n P^{r_1+\cdots+r_n}\\
&= T_j P^{r_1+\cdots+r_n}\frac{1}{2^n}(Q_1+Q_2)^n\\
&\qquad + S^j_n\left[R_1\cdots R_n\frac{1}{2^n}(Q_1+Q_2)^n + S_n\right]\\
&= T_{j+1}\frac{1}{2^n}(Q_1+Q_2)^n + S^{j+1}_n
\end{aligned}$$

where

$$T_{j+1} = T_j P^{r_1+\cdots+r_n} + S^j_n R_1\cdots R_n. \tag{iv}$$

THEOREM 4.1. *Let Assumption 4.1 hold. If P is a conservative operator such* *that P^k is ergodic for all k then*

$$\sup\{P^n(Q_2 - Q_1)f:\ -1 \leqq f \leqq 1\} \xrightarrow[n\to\infty]{} 0.$$

PROOF. Note first that if $-1 \leqq h \leqq 1$ then

$$P^{n+1}(Q_2 - Q_1)h = P^n(Q_2 - Q_1)Ph \leqq \sup\{P^n(Q_2 - Q_1)f:\ -1 \leqq f \leqq 1\}.$$

Thus the sequence of suprema is monotone and it is enough to establish convergence of some subsequence. Apply now (i) to the function 1: $S_n 1 = 1 - R_1 \cdots R_n 1$, or $S_n 1 \leqq 1$ but we do not have equality by part (c) of Assumption 4.1. Fix n, to be chosen later, and put $g = \lim_{j\to\infty} S_n^j 1$. Then $0 \leqq g \leqq 1$ and, by (i),

$$P^{r_1 + \cdots + r_n} g \geqq S_n g = g.$$

Since P^k is ergodic and conservative for all k, we must have $g = \text{const}$ and $R_1 \cdots R_n g = 0$ thus $g = 0$. Let us apply (iii) to the function 1: $1 = T_j 1 + S_n^j 1$ or $T_j 1 \leqq 1$ thus $\| T_j \| \leqq 1$. Use (iii) again for $-1 \leqq f \leqq 1$:

$$P^{(r_1 + \cdots + r_n)j}(Q_2 - Q_1)f = T_j \frac{1}{2^n}(Q_1 + Q_2)^n (Q_2 - Q_1)f + S_n^j(Q_2 - Q_1)f.$$

Now,

$$| S_n^j(Q_2 - Q_1)f | \leqq 2S_n^j 1 \xrightarrow[j\to\infty]{} 0$$

by the above remarks. On the other hand

$$\left\| T_j \frac{1}{2^n}(Q_1 + Q_2)^n (Q_2 - Q_1) \right\| \leqq \frac{1}{2^{n-1}} + \frac{1}{2^n} \sum_{k=0}^{n-1} \left| \binom{n}{k} - \binom{n}{k+1} \right|$$

by the binomial formula. The sequence $\binom{n}{k}$ increases for $0 \leqq k \leqq n/2$ and decreases for $n/2 \leqq k \leqq n$. Thus the sum is bounded by $2^{-n}\binom{n}{n/2}$ which is, by the Stirling Formula, $O(1/\sqrt{n})$ from which the theorem follows.

COROLLARY. *Let P^j be ergodic and conservative for all j. For a fixed k either*

$$\sup\{(P^{n+k} - P^n)f:\ -1 \leqq f \leqq 1\} = 2$$

for all n and almost all x (equivalently $P^n \wedge P^{n+k} = 0$) or

$$\lim_{n\to\infty} \sup\{(P^{n+k} - P^n)f:\ -1 \leqq f \leqq 1\} = 0.$$

PROOF. Put

$$h_n = \sup\{(P^{n+k} - P^n)f:\ -1 \leqq f \leqq 1\}$$

and note that:

(1) $0 \leqq h_n \leqq 2$: obvious.

(2) $h_{n+1} \leqq h_n$: if $-1 \leqq f \leqq 1$ then

$$(P^{n+1+k} - P^{n+1})f = (P^{n+k} - P^n)Pf \leqq h_n.$$

(3) $h_{n+1} \leqq Ph_n$: if $-1 \leqq f \leqq 1$ then

$$(P^{n+1+k} - P^{n+1})f = P(P^{n+k} - P^n)f \leqq Ph_n.$$

(4) $(P^n \wedge P^{n+k})1 = 1 - \frac{1}{2}h_n$:

$$\begin{aligned}(P^n \wedge P^{n+k})1 &= \inf\{P^n g + P^{n+k}(1-g) : 0 \leqq g \leqq 1\}\\ &= 1 - \sup\{(P^{n+k} - P^n)g : 0 \leqq g \leqq 1\}\\ &= 1 - \tfrac{1}{2}\sup\{(P^{n+k} - P^n)f : -1 \leqq f \leqq 1\}\end{aligned}$$

since $g = \frac{1}{2}(f+1)$ where $-1 \leqq f \leqq 1$. Note $h_n \equiv 2$ if and only if $P^n \wedge P^{n+k} = 0$. Let $h = \lim h_n$ (by (2)), then $Ph = h$ (by (3)) hence $h = \text{const} = \alpha$. If $\alpha = 2$ we are done; let $\alpha < 2$:

$$(P^n \wedge P^{n+k})1 = 1 - \frac{1}{2}h_n \uparrow 1 - \frac{\alpha}{2} > 0.$$

Apply Assumption 4.1 where $Q_1 = I$ and $Q_2 = P^k$ and $R_i = P^{n_i} \wedge P^{n_i+k}$ (the choice of n_i will be explained later). Now $P^{n_i+k} \geqq R_i$ and $P^{n_i+k} \geqq R_i P^k$ so we take $r_i = n_i + k$ and it remains to verify (c) of Assumption 4.1.

If $R_1 \cdots R_m 1 \neq 0$ for an appropriate choice of $n_1, \cdots, n_m$ then

$$R_1 \cdots R_m (P^n \wedge P^{n+k})1 \to \left(1 - \frac{\alpha}{2}\right) R_1 \cdots R_m 1 \neq 0$$

and $n = n_{m+1}$ may be chosen so that the left-hand side does not vanish.

Following D. Revuz put

$$G = \left\{k : \limsup_{n\to\infty} \{(P^{n+k} - P^n)f : -1 \leqq f \leqq 1\} = 0\right\}.$$

If $i, j \in G$ then $i + j \in G$. If $i + j \in G$ and $i \in G$ then $j \in G$:

$$P^{n+j} - P^n = (P^{n+i+j} - P^n) - (P^{n+i+j} - P^{n+j}).$$

Thus there exists an integer d such that $G = \text{multiples of } d$.

If $d \neq 0$ consider the subspace of L_1

$$K = \{u : \|uP^n\| \underset{n\to\infty}{\longrightarrow} 0\}.$$

If $u = v(P^d - I)$ then

$$\|uP^n\| = \|v(P^{n+d} - P^n)\| \leqq \int |v| \sup\{(P^{n+d} - P^n)f : -1 \leqq f \leqq 1\} d\lambda \underset{n\to\infty}{\longrightarrow} 0.$$

Hence $L_1(I - P^d) \subset K$ and by the Hahn Banach Theorem and since P^d is ergodic and conservative we have

$$\textit{if} \int u d\lambda = 0 \textit{ then } \|uP^n\| \underset{n\to\infty}{\longrightarrow} 0.$$

Let us assume a stronger version of Assumption 4.1 by taking $r_i = r$, $R_i = R$ (independent of i) and $R1 \geqq \text{const} > 0$.

ASSUMPTION 4.2. *Let P, Q_1 and Q_2 be commuting Markov operators such that*

(a) $P1 = Q_1 1 = Q_2 1 = 1$.

(b) *There exists an integer r and a Markov operator R such that*

$$P^r \geqq RQ_1 \quad \textit{and} \quad P^r \geqq RQ_2.$$

(c) $R1 \geqq \text{const} = \delta > 0$.

Since Assumption 4.2 implies Assumption 4.1 we immediately get:

$$\text{(i*)} \qquad P^{rn} = R^n \frac{1}{2^n}(Q_1 + Q_2)^n + S_n; \qquad S_n \geqq 0.$$

$$\text{(ii*)} \qquad S_{n+1} = R^n \tilde{S} \frac{1}{2^n}(Q_1 + Q_2)^n + S_n P^r.$$

$$\text{(iii*)} \qquad P^{rnj} = T_j \frac{1}{2^n}(Q_1 + Q_2)^n + S_n^j; \qquad T_j \geqq 0.$$

$$\text{(iv*)} \qquad T_{j+1} = T_j P^{rn} + S_n^j R^n.$$

As before $\|T_j\| \leqq 1$, from (i*) when applied to 1, follows

$$S_n 1 = 1 - R^n 1 \leqq 1 - \delta^n < 1.$$

Thus

$$\|P^{rnj}(Q_2 - Q_1)\| \leqq \left\| \frac{1}{2^n}(Q_1 + Q_2)^n (Q_2 - Q_1) \right\| + 2\|S_n\|^j$$

and the first term is $O(1/\sqrt{n})$ while the second term is bounded by $(1 - \delta^n)^j \underset{s\to\infty}{\rightarrow} 0.$

THEOREM 4.2. *Let Assumption* 4.2 *hold. Then*

$$\|P^n(Q_2 - Q_1)\| \xrightarrow[n\to\infty]{} 0.$$

PROOF. $\|P^n(Q_2 - Q_1)\|$ is monotone and the theorem follows from the above remarks.

COROLLARY. *Let P be a Markov operator with $P1 = 1$. Let k be a fixed integer, then either $\|P^{n+k} - P^n\| = 2$ for every n or $\lim_{n\to\infty}\|P^{n+k} - P^n\| = 0$.*

PROOF. Let $\|P^{m+k} - P^m\| < 2$ for some m. Put $R = P^m \wedge P^{m+k}$, $Q_1 = I$, $Q_2 = P^k$ and $r = m + k$. Note that $R1 = 1 - \frac{1}{2}\sup\{(P^{m+k} - P^m)f\colon\ -1 \leqq f \leqq 1\} \geqq 1 - \frac{1}{2}\|P^{m+k} - P^m\| > 0$.

Thus Assumption 4.2 holds and Theorem 4.2 applies.

As before, if $G = \{k : \|P^{n+k} - P^n\| \xrightarrow[n\to\infty]{} 0\}$ then G consists of multiples of a fixed integer.

Let us conclude this section with some results that will be useful for the study of Harris operators.

ASSUMPTION 4.3. *Let P, Q_1 and Q_2 be commuting Markov operators such that*

(a) $P1 = Q_11 = Q_21 = 1$.

(b) *There exist an integer r and a Markov operator R such that*

$$P^r \geqq Q_1R \quad \textit{and} \quad P^r \geqq Q_2R.$$

(c) $R1 = \text{const} = \delta > 0$.

Note that (c) is much stronger than (c) of Assumption 4.2.

The same argument as before will show

$$\text{(i**)} \qquad P^{rn} = \frac{1}{2^n}(Q_1 + Q_2)^n R^n + S_n; \qquad S_n \geqq 0.$$

$$\text{(ii**)} \qquad S_{n+1} = \frac{1}{2^n}(Q_1 + Q_2)^n \tilde{S} R^n + P^r S_n.$$

$$\text{(iii**)} \qquad P^{rnj} = \frac{1}{2^n}(Q_1 + Q_2)^n T_j + S_n^j; \qquad T_j \geqq 0.$$

$$\text{(iv**)} \qquad T_{j+1} = P^{rn} T_j + R^n S_n^j.$$

Apply (i**) to the function 1:

$$1 = \delta^n + S_n 1$$

(note we used equality: $R1 = \delta$). Now $T_1 1 = R^n 1 = \delta^n = \text{const} = \beta_1$. Let us prove, by induction, that $T_j 1 = \text{const} = \beta_j$: by (iv**) applied to 1

$$T_{j+1}1 = \beta_j + \delta^n(1-\delta^n)^j = \text{const} = \beta_{j+1}.$$

Apply now (iii**) to the function 1:

$$1 = \beta_j + S_n 1 \quad \text{or} \quad \beta_j \leqq 1$$

hence $\| T_j \| \leqq 1$. Hence we may use (iii**) to conclude

$$\|(Q_2 - Q_1)P^{mj}\| \leqq \left\| \frac{1}{2^n}(Q_2 - Q_1)(Q_1 + Q_2)^n \right\| + 2\|S_n\|^j$$

and as before

THEOREM 4.3. *Let Assumption* 4.3. *hold. Then*

$$\lim_{n\to\infty} \|(Q_2 - Q_1)P^n\| = 0.$$

COROLLARY. *Let P be a Markov operator with $P1 = 1$. Let μ be an invariant measure for P namely*:

$$0 \leqq \mu, \qquad \mu(X) = 1 \quad \textit{and} \quad \int (Pf)d\mu = \int f d\mu.$$

Put $Vf = \int f d\mu$, V is a Markov operator. Let there exist a measure τ satisfying $0 \leqq \tau$, $0 < \tau(X) \leqq 1$ and if $Rf = \int f d\tau$ then $P^r \geqq R$ for some integer r. Then

$$\lim_{n\to\infty} \|P^n - V\| = 0.$$

PROOF. Note first that $PV = VP = V^2 = V$. Also $P1 = V1 = 1$. Now $R1 = \tau(X) = \text{const}$. Also $P^r \geqq R = PR = VR$ and Assumption 4.3 holds for $Q_1 = P$ and $Q_2 = V$. Hence $\|(V-P)P^n\| \to 0$ by Theorem 4.3, but $(V-P)P^n = V - P^{n+1}$.

References. The Corollary of Theorem 4.1 is the "zero-two" law of Ornstein and Sucheston [20]. We followed here [6].

V. Harris Condition

Let us introduce some terminology.

A "density" is a function k on $X \times X$ such that $0 \leqq k(x,y)$ *and it is jointly measurable and* $\int k(x,y)\lambda(dy)$ *is a bounded function of* x.

The "integral kernel" of a density is given by

$$Kf(x) = \int k(x,y)f(y)\lambda(dy), \qquad f \in L_\infty.$$

If $K1 \leqq 1$ then K is a Markov operator. Now

$$\int_A K1_B d\lambda = \iint_{AB} k(x,y)\lambda(dy)\lambda(dx) = \tilde{K}(A \times B)$$

where $\tilde{K}$ is a measure on $(X \times X, \Sigma \times \Sigma)$ given by $d\tilde{K} = kd\lambda^2$.

Let P be any Markov operator and define $\tilde{P}$ on rectangles of $X \times X$ by

$$\tilde{P}(A \times B) = \int_A P1_B d\lambda.$$

In order to extend this set function, linearly, to the field of all finite unions of disjoint rectangles (see I.6 and III.1 of [16]) we have to show:

If $A \times B = \bigcup_{i=1}^n A_i \times B_i$ *where* $A_i \times B_i$ *are disjoint then*

$$\tilde{P}(A \times B) = \sum_{i=1}^n \tilde{P}(A_i \times B_i).$$

Now $A = \bigcap_{i=1}^n [A_i \cup (A - A_i)] = \bigcup E_j$ where the sets E_j are disjoint and for every i and j either $E_j \subset A_i$ or $E_j \cap A_i = \emptyset$. We may assume that $\lambda(E_j) > 0$ (discard the others). Now

$$1_A(x)1_B(y) = \sum_{i=1}^n 1_{A_i}(x)1_{B_i}(y),$$

multiply by $1_{E_j}(x)$ and evaluate at $x \in E_j$ to obtain

$$1_B = \sum_{i:\, E_j \subset A_i} 1_{B_i}; \qquad P1_B = \sum_{i:\, E_j \subset A_i} P1_{B_i}.$$

Thus

$$\int_{E_j} P1_B d\lambda = \sum_{i:\, E_j \subset A_i} \int_{E_j} P1_{B_i} d\lambda = \sum_i \int_{E_j \cap A_i} P1_{B_i} d\lambda.$$

Sum over j:

$$\int_A P1_B d\lambda = \sum_i \int_{A_i} P1_{B_i} d\lambda.$$

We shall use $\tilde{P}$ to denote the extension of P too.

LEMMA 5.1. *Let P be a Markov operator and K, K_1 and K_2 be integral kernels of the densities k, k_1 and k_2.*

(1) *If* $P \leqq K$ *then* P *is an integral kernel too.*

(2) $K_1 \leqq K_2$ *if and only if* $k_1 \leqq k_2$ *a.e.* λ^2.

(3) PK *and* KP *are integral kernels.*

(4) $K_1 \vee K_2$ $(K_1 \wedge K_2)$ *is an integral kernel whose density is* $\max(k_1, k_2)$ $(\min(k_1, k_2))$.

PROOF. (1) If $P \leqq K$ then $\tilde{P} \leqq \tilde{K}$. Now $\tilde{K}$ is continuous at $\varnothing$ (if $E_n \in \Sigma \times \Sigma$ and $E_n \downarrow \varnothing$ then $\tilde{K}(E_n) \to 0$). Thus $\tilde{P}$ is continuous at $\varnothing$ too and has a unique extension to all of $\Sigma \times \Sigma$. The extension will be denoted again by $\tilde{P}$. By the uniqueness of the extension of $\tilde{K} - \tilde{P}$ we must have $\tilde{P} \leqq \tilde{K}$ on all of $\Sigma \times \Sigma$. Thus $\tilde{P}$ is a measure (countably additive) and $\tilde{P} \ll \lambda^2$. If $d\tilde{P} = r d\lambda^2$ then $r \geqq 0$ is jointly measurable and

$$\int_A P1_B d\lambda = \int_A \int_B r(x, y)\lambda(dy)\lambda(dx)$$

hence $P1_B = \int_B r(x, y)\lambda(dy)$ and P is an integral kernel.

(2) Let $k_1 \leqq k_2$ a.e. λ^2, then for almost all x, $\lambda\{y : k_1(x, y) \leqq k_2(x, y)\} = 1$. Thus a.e. $K_1 1_A(x) \leqq K_2 1_A(x)$.

Conversely, let $K_1 \leqq K_2$ then $\tilde{K}_1 \leqq \tilde{K}_2$ on rectangles and, by unique extension, on $\Sigma \times \Sigma$ thus $k_1 \leqq k_2$ a.e. λ^2.

(3) Let $k_0 \equiv 1$ and K_0 be its integral kernel.

If $k \leqq$ const then $K \leqq$ const K_0 but $(PK_0)f = (\int f d\lambda)P1$: The integral kernel of the density $q(x, y) = P1(x)$.

$$(K_0P)f = \int (Pf)(y)\lambda(dy) = \int (1P)(y)f(y)\lambda(dy):$$

The integral kernel of the density $q(x, y) = (1P)(y)$.

By part (1) we have, if k is bounded, that PK and KP are integral kernels. If k is not bounded put $k_n = \min(k, n)$ and let K_n be its integral kernel. Let q_n be the density of K_nP (PK_n). The sequence q_n increases by part (2). Let $q \Leftarrow \lim q_n$. If $0 \leqq f \in L_\infty$ then

$$\int q(x, y)f(y)\lambda(dy) = \lim_{n \to \infty} \int q_n(x, y)f(y)\lambda(dy)$$

$$= \lim_{n \to \infty} (K_nP)f(x) = \lim_{n \to \infty} K_n(Pf)(x) = K(Pf)(x) = (KP)f(x).$$

(4) Note that $K_1 \vee K_2 \leqq K_1 + K_2$ thus, by part (1), it is an integral kernel. Let k_3 be the density of $K_1 \vee K_2$. By part (2), $k_3 \geqq \max(k_1, k_2) \geqq k_i$, $i = 1, 2$; again by

part (2) the integral kernel of $\max(k_1, k_2)$ dominates both K_1 and K_2 and thus $K_1 \vee K_2$. Use (2) again to conclude $\max(k_1, k_2) \geqq k_3$.

Let P be a Markov operator and put

$$\phi = \{k : k \text{ is the density of } K \text{ where } K \leqq P\}.$$

By part (4) of the previous lemma *if* $k_1, k_2 \in \phi$ *then* $\max(k_1, k_2) \in \phi$ *too*. Thus if $\alpha = \sup\{\iint k d\lambda^2 : k \in \phi\}$ then $\alpha = \lim \iint k_n d\lambda^2$ where $k_n \in \phi$ and $k_n \uparrow$. Put $q = \lim k_n$ and let Q be its integral kernel. By Fatou's Lemma $Q \leqq P$ or $q \in \phi$. Now $\iint q d\lambda^2 = \alpha$, thus if $k \in \phi$ then $\max(k, q) \in \phi$ and $\iint \max(k, q) d\lambda^2 \leqq \alpha = \iint q d\lambda^2$, thus q is the maximal element of ϕ.

THEOREM 5.2. *Every Markov operator P can be decomposed to $P = Q + R$ where Q is an integral kernel and R is a Markov operator that does not dominate any integral kernel.*

DEFINITION 5.1. The above decomposition will be called the Harris Decomposition. We shall denote the Harris Decomposition of P^n by $P^n = Q_n + R_n$.

DEFINITION 5.2. The Markov operator P is called a Harris operator if

(1) P is conservative.

(2) If $\lambda(A) > 0$ then $\sum_{n=1}^{\infty} Q_n 1_A$ is not identically zero.

Note (2) is equivalent to

(2') If $0 \leqq f \in L_\infty$ and $\sum_{n=1}^{\infty} Q_n f \equiv 0$ then $f \equiv 0$.

Recall Definitions 3.1, 3.2 and 3.3.

LEMMA 5.3. *If P is a Harris operator then $\Sigma^{(1)}$ is atomic.*

PROOF. Assume to the contrary, $A \in \Sigma^{(1)}$ and $A = \bigcup_{i=1}^{2^n} A_{i,n}$ where $A_{i,n} \in \Sigma^{(1)}$, $\lambda(A_{i,n}) = 2^{-n}\lambda(A)$ and $A_{j,n+1}$ are obtained by splitting each set $A_{i,n}$ into two sets, in $\Sigma^{(1)}$, of equal measure. The sets $A_{i,n}$, $1 \leqq i \leqq 2^n$ are disjoint, thus

$$\sum_{i=1}^{2^n} P^k 1_{A_{i,n}}(x) = P^k 1_A(x) \leqq 1,$$

and each term in the sum is either zero or one by the definition of $\Sigma^{(1)}$. Thus for all i, with at most one exception, $P^k 1_{A_{i,n}}(x) = 0$. The same holds for Q_k: For a fixed k and x

$$Q_k 1_A(x) = Q_k 1_{A_{j,n}}(x) = \int_{A_{j,n}} q_k(x, y)\lambda(dy)$$

where $j = j(x, n)$ and the sets $A_{j,n}$ decrease as n increases. Thus $Q_k 1_A(x) = 0$ and, by Definition 5.2, $\lambda(A) = 0$.

REMARK. The invariant sets are in $\Sigma^{(1)}$, hence if P is a Harris operator its collection of invariant sets is atomic and we may assume with no loss of generality that P is ergodic too.

Note that $\Sigma^{(2)}$ is atomic too thus, by Theorem 3.6, the restriction of P^d to one of the atoms of $\Sigma^{(2)}$ has ergodic powers.

LEMMA 5.4. *$Q_{n+m} \geqq P^n Q_m \geqq Q_n Q_m$; $Q_{n+m} \geqq Q_n P^m$ and $R_{n+m} \leqq R_n R_m$.*

PROOF. $P^{n+m} = (Q_n + R_n)(Q_m + R_m) = P^n Q_m + Q_n R_m + R_n R_m$. The first two terms are integral kernels by part (3) of Lemma 5.1.

If one chooses a particular version of $q_k(x, y)$ then $Q_k f(x)$ is defined at *every* point x, by $\int q_k(x, y) f(y) \lambda(dy)$, even if f is defined a.e. only. Thus $Q_k(P^m f)(x)$ is everywhere defined:

$$Q_k(P^m f)(x) = \int q_k(x, y)(P^m f)(y) \lambda(dy) = \int [q_k(x, \cdot) P^m](y) f(y) \lambda(dy).$$

Let us prove that $[q_k(x, \cdot) P^m](y)$ is the density of the integral kernel $Q_k P^m$. By the above remark it suffices to show that it is jointly measurable.

Consider the collection of densities $r(x, y)$ such that $[r(x, \cdot) P](y)$ is jointly measurable.

It is clear that this collection is linear and monotone, thus to show that every measurable function satisfies this condition it is enough to show that the characteristic function of every set in $\Sigma \times \Sigma$ does.

Let us study the collection of sets, E, in $\Sigma \times \Sigma$ such that $[1_E(x, \cdot) P](y)$ is jointly measurable.

Every rectangle has this property. It is a monotone collection.

By theorem I.4.2 of [16] every set in $\Sigma \times \Sigma$ has the desired property. Let us choose a version of q_n such that $Q_{n+1} f(x) \geqq Q_n(Pf)(x)$ at every point x if $0 \leqq f \in L_\infty$. Suppose we chose $q_1 \cdots q_k$, then $P^{k+1} = Q_k P + R_k P$. Choose the density of $Q_k P$ at every point as above and add to it the greatest integral kernel dominated by $R_k P$.

Therefore

LEMMA 5.5. *We may choose versions of the densities q_n in such a way that, if $0 \leqq f \in L_\infty$, then*

$$Q_{n+m} f(x) \geqq Q_n(P^m f)(x)$$

at every point x.

PROOF. We proved the lemma for $m = 1$. With this choice we have

$$Q_{n+m+1}f(x) \geqq Q_{n+m}(Pf)(x) \geqq Q_n(P^m Pf)(x)$$

by an induction argument on m.

Throughout the rest of the paper we shall assume that q_n satisfy Lemma 5.5.

THEOREM 5.6. *Let P be an ergodic and conservative Markov operator. P is a Harris operator if and only if $Q_k \neq 0$ for some k.*

If P is a Harris operator then

(a) $Q_n 1 \uparrow 1$ *a.e.*

(b) *If $\lambda(A) > 0$ then $\sum_{n=1}^{\infty} Q_n 1_A(x) = \infty$ at every point x for which* (a) *holds.*

PROOF. Let $Q_k \neq 0$ and $\lambda(A) > 0$. By Lemma 5.4

$$\sum_{n=1}^{\infty} Q_{k+n} 1_A \geqq Q_k \sum_{n=1}^{\infty} P^n 1_A \geqq Q_k 1 \neq 0.$$

Thus P is a Harris operator.

(a) $R_1 1 \leqq 1$ and, by Lemma 5.4,

$$R_{m+1} 1 \leqq R_m R_1 1 \leqq R_m 1.$$

Put $g = \lim_{m\to\infty} R_m 1$, then $0 \leqq g \leqq 1$ and

$$P^k g = Q_k g + R_k g \geqq \lim_{m\to\infty} R_k R_m 1 \geqq \lim_{m\to\infty} R_{k+m} 1 = g.$$

Thus, since P^k is conservative, $P^k g = g$ and $Q_k g = 0$, by Harris Condition $g = 0$.

(b) Let $Q_m 1(x) > 0$ (by (a)), then

$$\sum_{n=1}^{\infty} Q_{m+n} 1_A(x) \geqq Q_m \left(\sum_{n=1}^{\infty} P^n 1_A \right)(x).$$

Now, $\sum_{n=1}^{\infty} P^n 1_A \geqq N1$ for every constant N thus

$$\sum_{n=1}^{\infty} Q_n 1_A(x) \geqq N Q_m 1(x)$$

and the left-hand side must be infinite.

References. The idea of the decomposition goes back to W. Doeblin [3]. A similar idea may be found in [21]. Our description is closer to Harris's [9]. Lemma 5.3 is due to J. Feldman [4].

In our study we do not assume that P is given by a transition probability and that Σ is separable, which complicates the arguments considerably.

See also [17] and [18].

VI. Harris Lemma

Let P be an ergodic Harris operator and let $Q_k \neq 0$. Choose $\varepsilon > 0$ so that $0 < \lambda^2(\{(x,y): q_k(x,y) > \varepsilon\})$.

Thus $0 < \int \lambda(\{y : q_k(x,y) > \varepsilon\})\lambda(dx)$ and the integrand is greater than $\delta > 0$ on a set E with $\lambda(E) > 0$. Let $\lambda(A) > 1 - \delta/2$ and $x \in E$ then

$$\lambda(A' \cup \{y: q_k(x,y) \leqq \varepsilon\}) \leqq \frac{\delta}{2} + 1 - \delta = 1 - \frac{\delta}{2}$$

or

$$\lambda(A \cap \{y: q_k(x,y) > \varepsilon\}) \geqq \frac{\delta}{2}.$$

Thus

$$P^k 1_A(x) \geqq \int_A q_k(x,y)\lambda(dy) \geqq \varepsilon\lambda(A \cap \{y: q_k(x,y) > \varepsilon\}) \geqq \frac{1}{2}\varepsilon\delta.$$

Let B be any set with $\lambda(B) > 0$. Choose N so large that if $A = \{x: \Sigma_{n=1}^N P^n 1_B(x) \geqq 1\}$ then $\lambda(A) > 1 - \delta/2$. Now $1_A \leqq \Sigma_{n=1}^N P^n 1_B$ hence

$$\sum_{n=1}^{N} P^{k+n} 1_B \geqq P^k 1_A \geqq \frac{1}{2}\varepsilon\delta 1_E.$$

Let us summarize.

LEMMA 6.1. *Let P be an ergodic Harris operator. There exists an integer k, two positive constants ε and δ, and a set E with $\lambda(E) > 0$ such that*:

(a) *If $x \in E$ then $\lambda(\{y: q_k(x,y) > \varepsilon\}) > \delta$.*

(b) *If $\lambda(A) > 1 - \delta/2$ then $P^k 1_A \geqq \frac{1}{2}\varepsilon\delta 1_E$.*

(c) *If $\lambda(B) > 0$ then there exists an integer $N = N(B)$ such that $\Sigma_{n=0}^N P^n 1_B \geqq \frac{1}{2}\varepsilon\delta 1_E$.*

DEFINITION 6.1. A set E is called "reserve" if $\lambda(E) > 0$ and for every set A with $\lambda(A) > 0$ there exists an $\varepsilon = \varepsilon(A) > 0$ and an integer $N = N(A)$ such that $\Sigma_{n=0}^N P^n 1_A \geqq \varepsilon 1_E$.

Lemma 6.1 shows that an ergodic Harris operator has reserve sets. Let E be reserve and put

$$E_K = \left\{x: \sum_{k=0}^{K} P^k 1_E(x) \geqq 1\right\}.$$

Then

$$1_{E_K} \leqq \sum_{k=0}^{K} P^k 1_E.$$

If $\lambda(A)>0$ find $\varepsilon>0$ and an integer N such that $\sum_{n=0}^{N} P^n 1_A \geqq \varepsilon 1_E$, hence

$$\varepsilon 1_{E_K} \leqq \varepsilon \sum_{k=0}^{K} P^k 1_E \leqq \sum_{k=0}^{K} \sum_{n=0}^{N} P^{k+n} 1_A \leqq (K+1) \sum_{j=0}^{N+K} P^j 1_A$$

since each power $k+n$ repeats itself at most $K+1$ times. Thus E_k is reserve again and $E_K \uparrow X$.

THEOREM 6.2. *Let P be an ergodic Harris operator, then there exist reserve sets E_K with $E_K \uparrow X$.*

In the rest of this section we shall use the "zero-two" law for Harris operators.

LEMMA 6.3. *Let P be a Harris operator such that P^n is ergodic for every n. For every fixed integer k, $\bigcup_{n=1}^{\infty}\{(x,y): q_{nk}(x,y)>0\} = X \times X$ a.e. λ^2.*

PROOF. If $Q_j \neq 0$ then $Q_{j+r} \geqq Q_j P^r$ hence $Q_{j+r} \neq 0$ too. Thus P^k is again a Harris operator with ergodic iterates. Let us then prove the lemma for $k=1$. By Theorem 5.6 there exists a set Y with $\lambda(Y)=0$ such that if $\lambda(A)>0$ and $x_0 \notin Y$ then $\sum_{n=1}^{\infty} Q_n 1_A(x_0) = \infty$. Note we assumed, as in Lemma 5.5, that $Q_n f(x)$ is everywhere defined. Let us study the following situation:

$$(*) \qquad x_0 \notin Y \quad \textit{and} \quad q_n(x_0, y) = 0 \quad \textit{for all } n.$$

By Lemma 5.4 (*) implies

$$0 = \int q_n(x_0, z) q_m(z, y) \lambda(dz) = \int_{A_m} q_n(x_0, z) q_m(z, y) \lambda(dz)$$

where $A_m = \{z : q_m(z,y) > 0\}$. Thus $q_n(x_0, z) = 0$ for almost all $z \in A_m$ or $Q_n 1_{A_m}(x_0) = 0$. Since $x_0 \notin Y$ we must have $\lambda(A_m) = 0$. Thus (*) implies

$$(**) \qquad q_n(z, y) = 0 \quad \textit{for almost all } z.$$

Given any set A we have

$$(\lambda Q_n)(A) = \int_X \int_A q_n(z,y)\lambda(dy)\lambda(dz) = \int_A \left[\int_X q_n(z,y)\lambda(dz)\right]\lambda(dy)$$

or

$$\frac{d(\lambda Q_n)}{d\lambda}(y) = \int_X q_n(z,y)\lambda(dz)$$

and if (**) holds then

$$\frac{d(\lambda Q_n)}{d\lambda}(y) = 0.$$

Let

$$E = \{(x, y)\colon q_n(x, y) = 0 \text{ for all } n \text{ and } x \not\in Y\}.$$

For a fixed $x_0 \not\in Y$, $E_{x_0} = \{y : q_n(x_0, y) = 0$ for all $n\}$, then

$$(\lambda Q_n)(E_{x_0}) = \int_{E_{x_0}} \frac{d(\lambda Q_n)}{d\lambda}(y)\lambda(dy) = 0$$

or, by Theorem 5.6, $\lambda(E_{x_0}) = 0$ and $\lambda^2(E) = \int \lambda(E_x)\lambda(dx) = 0$.

THEOREM 6.4. *Let P be a Harris operator such that P^n is ergodic for all n. Then*

$$\lim_{n\to\infty} \sup\{(P^{n+1} - P^n)f\colon -1 \leqq f \leqq 1\} = 0.$$

PROOF. Let $q_r \not\equiv 0$ and choose $k > r$. By the previous lemma

$$\lambda^2(\{(x, y)\colon q_{nk}(x, y) > 0\} \cap \{(x, y)\colon q_r(x, y) > 0\}) \neq 0$$

for some n, thus $Q_{nk} \wedge Q_r \neq 0$ hence $P^{nk} \wedge P^r \neq 0$ too and, by the Corollary to Theorem 4.1, we have

$$\lim_{m\to\infty} \sup\{(P^{m+j} - P^m)f\colon -1 \leqq f \leqq 1\} = 0$$

for some integer j. Let d be the smallest such integer; if $d \neq 1$ then $P^{md+1} \wedge P^{nd} = 0$ for every n and m. Thus $Q_{md+1} \wedge Q_{nd} = 0$ too.

Fix m so that $Q_{md+1} \neq 0$. By (4) of Lemma 5.1 we have $\min(q_{md+1}, q_{nd}) = 0$ for all n which contradicts Lemma 6.3.

References. Lemma 6.1 was proved by Harris in [9].

The notion of reserve sets was defined in [1].

Theorem 6.4 was proved by very different methods in [13].

VII. The induced operator

Let P be a Markov operator and E a fixed set with $\lambda(E) > 0$. For every set A with $\lambda(A) > 0$ put $T_A f = 1_A f$; then T_A is a Markov operator.

DEFINITION 7.1. $P_E = \sum_{n=0}^{\infty} (PT_{E'})^n PT_E$.

Note that

$$\sum_{n=0}^{N} (PT_{E'})^n PT_E 1 = \sum_{n=0}^{N} (PT_{E'})^n (P - PT_{E'})1$$

$$\leqq \sum_{n=0}^{N} (PT_{E'})^n 1 - \sum_{n=1}^{N+1} (PT_{E'})^n 1 = 1 - (PT_{E'})^{N+1} \leqq 1.$$

Thus $P_E 1 \leqq 1$ and P_E is a nonnegative linear contraction on L_∞. Now P_E is a Markov operator since if $0 \leqq f \leqq M$ then

$$P_E f \leqq \sum_{n=0}^{N} (PT_{E'})^n PT_E f + M \sum_{n=N+1}^{\infty} (PT_{E'})^n PT_E 1,$$

hence part (3) of Definition 1.1 holds too. The operator P_E is a Markov operator on $L_\infty(X, \Sigma, \lambda)$, but we shall use the operator $T_E P_E$ on $L_\infty(E, \Sigma_1, \lambda_1)$ where Σ_1 contains all measurable subsets of E and for $A \subset E$, $\lambda_1(A) = \lambda(A)/\lambda(E)$.

LEMMA 7.1. *Let $0 \leqq f \in L_\infty$ and $f = T_E f$, then*

$$\sum_{n=0}^{N} P^n f \leqq \sum_{n=0}^{N} P_E^n f$$

and

$$\sum_{n=0}^{N} T_E P^n f \leqq \sum_{n=0}^{N} (T_E P_E)^n f.$$

PROOF. The second inequality follows from the first, since $P_E = P_E T_E$ thus $(T_E P_E)^n = T_E P_E^n$. Now $P_E = PT_E + (PT_{E'})P_E$, hence $P = PT_E + PT_{E'} = P_E + (PT_{E'})(I - P_E)$. Let us prove the inequality by induction:

$$\begin{aligned}\sum_{n=0}^{N+1} P^n f &= f + P \sum_{n=0}^{N} P^n f \\ &\leqq f + P \sum_{n=0}^{N} P_E^n f \\ &= f + (P_E + PT_{E'}(I - P_E)) \sum_{n=0}^{N} P_E^n f \\ &= \sum_{n=0}^{N+1} P_E^n f + PT_{E'}(f - P_E^{N+1} f).\end{aligned}$$

Now $T_{E'} f = 0$ and the lemma follows.

COROLLARY. *If P is ergodic and conservative then so is $T_E P_E$, hence $T_E P_E 1_E = 1_E$.*

DEFINITION 7.2. A set E is called "special" if $\lambda(E) > 0$ and for every $A \subset E$ with $\lambda(A) > 0$ there exists an integer $N = N(A)$ and a constant $\varepsilon = \varepsilon(A)$ where $\varepsilon > 0$ and

$$\sum_{n=0}^{N} P_E^n 1_A \geqq \varepsilon 1_E.$$

By Lemma 7.1 and Theorem 6.2:

THEOREM 7.2. *Let P be an ergodic Harris operator, then there exist special sets E_K with $E_K \uparrow X$.*

THEOREM 7.3. *Assume $\lambda(E) > 0$ and $\Sigma_{k=0}^{\infty} P^k 1_E(x) > 0$ at every point. If η is a finite measure such that $\eta = \eta T_E$ and $\eta = \eta T_E P_E$ then $\mu = \Sigma_{n=0}^{\infty} \eta (PT_{E'})^n$ is σ finite and satisfies $\mu P = \mu$.*

REMARK. The assumption holds if P is ergodic and conservative.

PROOF.

$$\mu P = \mu PT_E + \mu PT_{E'} = \sum_{n=0}^{\infty} \eta (PT_{E'})^n PT_E + \sum_{n=1}^{\infty} \eta (PT_{E'})^n = \eta P_E + \mu - \eta = \mu$$

since $\eta P_E = \eta T_E P_E = \eta$.

Now $\int P^k 1_E d\mu = \int 1_E d\mu = \eta(E) = 1$, hence

$$\mu\left(\left\{x : P^k 1_E(x) \geqq \frac{1}{n}\right\}\right) < \infty$$

and μ is σ finite since $\Sigma_{k=0}^{\infty} P^k 1_E > 0$ implies

$$\bigcup_{k,n} \left\{x : P^k 1_E(x) \geqq \frac{1}{n}\right\} = X.$$

Note that $\mu T_E = \eta$.

The next result will be very useful in Section VIII.

THEOREM 7.4.

$$T_E - T_E P_E = (I - P) \sum_{n=0}^{\infty} (T_{E'} P)^n T_E$$

and

$$\sum_{n=0}^{\infty} (T_{E'} P)^n T_E 1 \leqq 1.$$

PROOF.

$$\sum_{n=0}^{N} (T_{E'} P)^n T_E 1 = T_E 1 + T_{E'} \sum_{n=0}^{N-1} (PT_{E'})^n PT_E 1 \leqq T_E 1 + T_{E'} 1 \leqq 1.$$

Now

$$T_E \sum_{n=0}^{N} (PT_{E'})^n PT_E = (I - T_{E'}) P \sum_{n=0}^{N} (T_{E'} P)^n T_E$$

$$= P \sum_{n=0}^{N} (T_{E'} P)^n T_E - \sum_{n=1}^{N+1} (T_{E'} P)^n T_E$$

$$= (P - I) \sum_{n=0}^{N} (T_{E'}P)^n T_E + T_E - (T_{E'}P)^{N+1} T_E.$$

Now $(T_{E'}P)^{N+1}T_E f \to 0$ for every $f \in L$ by the first part. Let $N \to \infty$ to conclude

$$T_E P_E = (P - I) \sum_{n=0}^{\infty} (T_{E'}P)^n T_E + T_E.$$

References. The definition of P_E and its use for finding a σ finite invariant measure were suggested by P. Halmos [8]. We followed here Harris [9] for Theorem 7.3. Theorem 7.4 was proved by S. Horowitz [11] and A. Brunel [1]. The notion of "special" sets was introduced in [17].

VIII. The Ornstein–Métivier–Brunel Theorem

ASSUMPTION 8.1. *The operator P is ergodic and conservative and E is a special set for P.*

REMARK. If P is an ergodic Harris operator then, by Theorem 7.2, we may choose E to be very close to X.

DEFINITION 8.1. $S = \Sigma_{n=0}^{\infty}(1/2^{n+1})T_E P_E^n$ on $L_\infty(E, \Sigma_1, \lambda_1)$.

By the Corollary of Lemma 7.1, $S1_E = 1_E$.

LEMMA 8.1. *Assume* 8.1. *If* $A \subset E$ *and* $\lambda_1(A) > 0$ *then* $S1_A \geqq \varepsilon 1_E$ *where* $\varepsilon = \varepsilon(A) > 0$.

PROOF. $S1_A \geqq (1/2^N)\Sigma_{n=0}^{N} T_E P_E^n 1_A \geqq (1/2^N)\varepsilon(A)1_E$ if $N = N(A)$, by Definition 7.2.

From the lemma follows that

$$\liminf \int S^n 1_A d\lambda_1 \geqq \varepsilon(A) > 0.$$

Here we deviate from our custom and quote, but not prove, theorem B of chapter IV of [5], to conclude.

There exists a finite invariant measure η, *for S.*

Now $\eta = \eta T_E$ and

$$0 = \eta(T_E - S) = \left[\eta \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}(T_E + \cdots + T_E P_E^{n-1})\right](T_E - T_E P_E),$$

thus by replacing η by the term in brackets we may and shall assume

(*) $\eta = \eta T_E = \eta T_E P_E = \eta S$ *and* η *is finite.*

Put $\mu = \Sigma_{n=0}^{\infty} \eta (PT_{E'})^n$ and recall Theorem 7.3.

(**) $0 \leqq \mu$ *is* σ *finite,* $\mu P = \mu$ *and* $\mu T_E = \eta$.

Let us improve Lemma 8.1.

LEMMA 8.2. *Assume* 8.1. *There exists a constant* $\alpha > 0$ *such that if* $A \subset E$ *and* $\lambda_1(A) \geqq 1 - \alpha$ *then* $S1_A \geqq \alpha 1_E$.

PROOF. Assume, to the contrary, that for each $j \geqq 2$ we may find a set $A_j \subset E$ with $\lambda_1(A_j) \geqq 1 - 1/2^j$ but $S1_{A_j}(x) < 1/2^j$ on the set B_j with $\lambda_1(B_j) > 0$. Now $\Sigma_{j=2}^{\infty} \lambda_1(A'_j) \leqq \frac{1}{2}$ so if $A = \bigcap_{j=2}^{\infty} A_j$ then $\lambda_1(A) > \frac{1}{2}$, but $S1_A(x) \leqq S1_{A_j}(x) \leqq 1/2^j$ if $x \in B_j$, which contradicts Lemma 8.

For the next few results we shall refer to the Harris decomposition of S^n on $L_\infty(E, \Sigma_1, \lambda_1)$.

DEFINITION 8.2. The Harris decomposition is denoted by

$$S^n = T_n + U_n$$

where T_n is the maximal integral kernel and all operators are on $L_\infty(E, \Sigma_1, \lambda_1)$.

LEMMA 8.3. *Assume* 8.1, *then* $T_1 1_E \geqq \frac{1}{2}\alpha 1_E$.

PROOF. Let $K_0 f(x) = \int f(y)\lambda(dy)$ then, by Lemma 5.1 part (1), $S \wedge K_0$ is an integral kernel dominated by S, thus

$$T_1 1_E \geqq (S \wedge K_0) 1_E \geqq Sg + \int (1 - g) d\lambda_1$$

if $0 \leqq g \leqq 1_E$. Let $A = \{x : g(x) \geqq \frac{1}{2}\} \cap E$. Then $g \geqq \frac{1}{2} 1_A$ and on A', $1 - g > \frac{1}{2}$ so $T_1 1_E \geqq \frac{1}{2} S 1_A + \frac{1}{2}\lambda_1(A')$. If $\lambda_1(A') \geqq \alpha$ then $T_1 1_E \geqq \alpha/2$, while if $\lambda_1(A') < \alpha$ then $\lambda_1(A) \geqq 1 - \alpha$ and $T_1 1_E \geqq \frac{1}{2} S 1_A \geqq \frac{1}{2}\alpha 1_E$ by Lemma 8.2.

LEMMA 8.4. *Assume* 8.1, *then* $T_2 1_A(x) \geqq \frac{1}{2}\varepsilon(A)\alpha 1_E(x)$ *outside of a fixed set (independent of* A*) of measure zero.*

PROOF. By Lemma 5.5, we have at *every* point

$$T_2 1_A(x) \geqq T_1(S1_A)(x) \geqq \varepsilon(A) T_1 1_E(x).$$

Apply now Lemma 8.3.

LEMMA 8.5. *Assume* 8.1. *If* k_2 *is the density of* T_2 *then* $k_2 > 0$ *a.e.* λ^2.

PROOF. Assume, to the contrary, that $\lambda^2(F)>0$ where $F=\{(x,y)\colon k_2(x,y)=0\}$. Then $\lambda(F_x)>0$ ($F_x=\{y\colon (x,y)\in F\}$) for a set of x's of positive measure. Thus $T_2 1_{F_x}(x)=0$ on a set of positive measure which contradicts Lemma 8.4.

Denote for $x\in E$ and $y\in E$

$$f_k(x)=\lambda_1\left(\left\{z\colon k_2(x,z)\geqq\frac{1}{k}\right\}\right),$$

$$g_k(y)=\lambda_1\left(\left\{z\colon k_2(z,y)\geqq\frac{1}{k}\right\}\right).$$

Now

$$1=\lambda_1^2(E\times E)=\int\lambda_1(\{z\colon k_2(x,z)>0\})\lambda_1(dx)$$

so $\lambda_1(\{z\colon k_2(x,z)>0\})=1$ for almost all x. Thus $f_k(x)\uparrow 1$ a.e. and similarly $g_k(y)\uparrow 1$ a.e. Hence:

LEMMA 8.6. *There exists an integer k such that the sets $F=\{x\colon f_k(x)\geqq 3/4\}$ and $G=\{y\colon g_k(y)\geqq 3/4\}$ have positive measures.*

If $x\in F$ and $y\in G$ then

$$\lambda\left(\left\{z\colon k_2(x,z)<\frac{1}{k}\right\}\cup\left\{z\colon k_2(z,y)<\frac{1}{k}\right\}\right)<\frac{1}{4}+\frac{1}{4}=\frac{1}{2}.$$

Thus

$$\lambda\left(\left\{z\colon k_2(x,z)\geqq\frac{1}{k}\right\}\cap\left\{z\colon k_2(z,y)\geqq\frac{1}{k}\right\}\right)>\frac{1}{2}$$

therefore

$$\int k_2(x,z)k_2(z,y)\lambda(dz)\geqq\frac{1}{2k^2}.$$

LEMMA 8.7. *Assume 8.1. If k_4 is the density of T_4 then $k_4(x,y)\geqq\beta 1_F(x)1_G(y)$ where $\beta>0$.*

PROOF.

$$k_4(x,y)\geqq\int k_2(x,z)k_2(z,y)\lambda(dz)\geqq\frac{1}{2k^2}\qquad\text{if } x\in F \text{ and } y\in G.$$

LEMMA 8.8. *Assume 8.1. Let τ be the measure on Σ_1 given by $\tau(A)=\beta\varepsilon(F)\lambda_1(G\cap A)$ then for every $0\leqq g\in L_\infty$*

$$S^5g\geqq\int g\,d\tau.$$

PROOF. $S^5 g \geqq ST_4 g \geqq S(\beta \int_G g d\lambda_1 1_F)$ by Lemma 8.7. Now, by Lemma 8.1, $S1_F \geqq \varepsilon(F)1_E$ or $S^5 g \geqq \beta\varepsilon(F)\int_G g d\lambda_1$ but $\int_G g d\lambda_1 = \int g d\tau$.

We may use now the Corollary to Theorem 4.3.

THEOREM 8.9. *Assume* 8.1 *and put* $Vf = \int_E f d\mu = \int_E f d\eta$ *(as defined in* (*) *and* (**)) *then*

$$\lim_{n\to\infty} \| S^n - V \| = 0.$$

Moreover there exists an integer n such that

$$\left\{ f\colon f \in L_\infty(X), f \text{ supported on } E \text{ and} \int f d\mu = 0 \right\} \subset \text{Range}(T_E - S^n).$$

PROOF. We need to prove the second part only. Let $\|S^n - V\| < 1$. The operator $T_E - V$ is a projection of $L_\infty(E)$ onto $\{f\colon f \in L_\infty(E) \text{ and } \int f d\mu = 0\}$ and it commutes with S, thus the range of $T_E - V$ is invariant under S and the norm of S^n restricted to this range is smaller than 1. Thus $T_E - S^n$ is invertible on $\{f\colon f \in L_\infty(E) \text{ and } \int f d\mu = 0\}$.

Note now

$$\text{Range}(T_E - S^n) \subset \text{Range}(T_E - S)\colon$$

$$T_E - S^n = (T_E - S)(T_E + S + \cdots + S^{n-1}).$$

Also

$$\text{Range}(T_E - S) \subset \text{Range}(T_E - T_E P_E)\colon$$

$$T_E - S = (T_E - T_E P_E) \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} (T_E + \cdots + T_E P_E^{n-1}).$$

Finally, by Theorem 7.4,

$$\text{Range}(T_E - T_E P_E) \subset \text{Range}(I - P).$$

Now if $f \in \text{Range}(I - P)$ then $f = (I - P)h$ for some $h \in L_\infty$ and

$$\left| \sum_{n=0}^{N} P^n f \right| = | h - P^{N+1} h | \leqq 2\| h \|.$$

Thus

THEOREM 8.10 (Ornstein–Métivier–Brunel Theorem). *Assume* 8.1 *and let* μ *be the invariant* σ *finite measure of P which is finite on E (as in* (**)). *If* $f \in L_\infty(X)$ *and is supported on E and* $\int f d\mu = 0$ *then* $|\sum_{n=0}^{N} P^n f| \leqq \text{const} < \infty$.

References. Results similar to Lemma 8.8 (and previous lemmas) were proved by Orey [18] and Neveu [17]. Theorem 8.10 was proved for random walks by Ornstein [11]. Métivier extended this result to Harris operators in [14]; he did not use special sets but a more restrictive class. The general result was proved by Brunel [1]. A very elegant proof, using the notion of quasicompact operators, was given by Horowitz [12]. Another presentation is given in [2]. In [7] Ghoussoub showed that the special sets are the largest class for which the Theorem holds.

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